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Correlators in AdS_3 string theory

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Abstract: The computation of two and three point functions in the Coulomb gas free field approach to string theory in the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ black hole background is reviewed. An interesting relation arises when comparing the results obtained using two different screening operators. The formalism is then modified to study string theory propagating in AdS_3 which is considered as the direct product of the $\mathrm{SL}(2)/\mathrm{U}(1)$ coset times a timelike free boson. This representation allows to naturally include the spectral flow symmetry and winding number in vertex operators and correlation functions. Two and three point tachyon amplitudes are computed in this new scenario and the results coincide with previous reports in the literature. Novel expressions are found for processes violating winding number conservation.

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1. Introduction

String theory on three dimensional anti de Sitter space (AdS_3) is an interesting model to analyse the AdS/CFT correspondence beyond the field theory approximation. Much progress has been achieved in understanding this theory in recent years (see [1, 2]), although there are still important issues to be clarified.

Unitarity has been the leading subject in this story during the last decade since, unlike string theory in flat spacetime, the Virasoro constraints seemed unable to annihilate all the negative norm states of the string propagating in AdS_3. Fortunately a naturally unitary spectrum has been revealed by the spectral flow symmetry disclosed in reference [3]. The new representations obtained by the spectral flow were originally considered in [2] in the context of string theory in the $\mathrm{SL}(2,\mathbb{R})$ group manifold. It was shown in reference [3] that they resolve some of the longstanding negative consequences of arbitrarily truncating the spin j (equivalently the mass) of the physical states when the target space is the universal cover of $\mathrm{SL}(2,\mathbb{R})$. This represents an important step in the construction of a consistent model, but the consistency of the theory cannot be completely established until interactions are included and the closure of the operator product expansion is determined. Indeed, regarded as a conformal field theory, string theory in AdS_3 will be completly characterized by the spectrum and the full set of three point functions.

In this paper we continue the study of correlation functions of string theory in AdS_3 which was started in reference [4]. Based on the proposal in [3] we consider the theory as the tensor product of the coset space $H_3^+/\,\mathrm{U}(1)$ (the euclidean version of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ group manifold) times the state space of a timelike free boson. The vertex operators are constructed and correlation functions are computed extending to the non-compact case a prescription developed for SU(2) in reference [5]. This formalism is suitable to manifestly include the spectral flow parameter or winding number ω. We explicitly construct two and three point functions of physical states using the modified Coulomb gas formalism developed in [4], and we then compare the expressions obtained with results reported earlier in the literature which were found by other methods.

Actually, various approaches have been followed to construct correlation functions in the $\mathrm{SL}(2,\mathbb{R})$ WZW model (the lagrangian formalism [6, 7], the bootstrap method [8, 9], the free field approximation [10, 11, 12]). The results obtained in these references were shown to agree in [12], where it is argued furthermore that the full interacting theory might be reducible to a free theory based on the precise equivalence of two, three and certain four point functions. Here we confirm that this is indeed the case for correlation functions of two and three tachyon states, using a different free field approach.

The plan of the article is as follows. In section 2 we review the computation of two and three point tachyon amplitudes in the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ two dimensional black hole background performed in reference [11] using the Coulomb gas free field approximation. As it is well known, there are two screening operators in this formalism [13, 14], so we take the opportunity to compare the results obtained using each of them. This leads to an interesting relation, similar to one recently proposed in Liouville theory [15]. In section 3 we recompute the two and three point tachyon amplitudes using a modified Coulomb gas prescription, suitable to deal with string theory in AdS_3. The results are shown to agree with previous reports in the literature for processes conserving winding number. However, novel results are obtained for three point functions violating winding number conservation by one unit. A summary of the results accomplished and conclusions are contained in section 4.

2. Free field approach to $\mathrm{SL}(2,\mathbb{R})$ WZW model

This section contains a review of the computation of correlation functions in $\mathrm{SL}(2,\mathbb{R})$ CFT using the Coulomb gas prescription and a detailed analysis of the ingredients that are needed in the standard formalism. Since we are ultimately interested in the application of the techniques to string theory in AdS_3, we start by briefly recalling the free field formulation of this model.

String theory on AdS_3 is described by the lagrangian

$$L = k(\partial\phi\bar{\partial}\phi + e^{2\phi}\bar{\partial}\gamma\partial\bar{\gamma})\,, \tag{2.1}$$

where $(\phi,\gamma,\bar{\gamma})$ are coordinates on the Euclidean AdS_3 spacetime, which is equivalent to the quotient space $H_3^+ = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$, and k is a constant related to the radius of curvature of spacetime l and the fundamental string length l_s, as $k = l^2/l_s^2$. Conformal invariance of the sigma model on this background requires in addition a NS-NS antisymmetric tensor field.

It is convenient to rewrite the action adding one-form auxiliary fields β, $\bar\beta$ as

$$L = k\left(\partial\phi\bar\partial\phi + \beta\bar\partial\gamma + \bar\beta\partial\bar\gamma - e^{-2\phi}\beta\bar\beta\right), \tag{2.2}$$

which gives (2.1) after integrating out β, $\bar\beta$.

This theory is a SL(2) WZW model with a current algebra specified by the following OPE

$$\begin{aligned} J^+(z)J^-(w) &= \frac{k}{(z-w)^2} - \frac{2}{(z-w)}J^3(w) + \cdots \\ J^3(z)J^\pm(w) &= \pm\frac{1}{(z-w)}J^\pm(w) + \cdots \\ J^3(z)J^3(w) &= \frac{-k/2}{(z-w)^2} + \cdots \end{aligned} \tag{2.3}$$

which can be realized in terms of the three fields β, γ, ϕ introduced above. These fields have correlators given by

$$\langle\beta(z)\gamma(w)\rangle = \frac{1}{z-w}; \qquad \langle\phi(z)\phi(w)\rangle = -\ln(z-w)\,. \tag{2.4}$$

There are also $\bar z$ dependent antiholomorphic fields $(\bar\beta(\bar z),\ \bar\gamma(\bar z),\ \phi(\bar z))$. However we shall discuss the left moving part of the theory only and assume that all the steps go through to the right moving part as well, indicating the left-right matching conditions where necessary.

The SL(2) currents can be thus represented as

$$\begin{aligned} J^+(z) &= \beta \\ J^3(z) &= -\beta\gamma - \frac{\alpha_+}{2}\partial\phi \\ J^-(z) &= \beta\gamma^2 + \alpha_+\gamma\partial\phi + k\partial\gamma\,, \end{aligned} \tag{2.5}$$

where $\alpha_+ = \sqrt{2(k-2)}$ and k is the level of the SL(2) algebra. They give rise to the Sugawara stress-energy tensor

$$T_{\mathrm{SL}(2)}(z) = \beta\partial\gamma - \frac{1}{2}\partial\phi\partial\phi - \frac{1}{\alpha_+}\partial^2\phi\,, \tag{2.6}$$

which leads to the following central charge of the Virasoro algebra

$$c = 3 + \frac{12}{\alpha_+^2} = \frac{3k}{k-2}\,. \tag{2.7}$$

The complete action associated with the energy-momentum tensor (2.6) is

$$S = \int d^2z \left(\frac{1}{2} \partial\phi\bar{\partial}\phi - \frac{2}{\alpha_+} R\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} \right) - S_{\text{int}} \,, \tag{2.8}$$

where the linear dilaton term can be interpreted as the effect of a background charge at infinity and $S_{\text{int}} = \int d^2z \beta\bar{\beta}e^{-2\phi}$. Notice that when $\phi \to \infty$ the interaction term vanishes and the theory can be treated perturbatively. The boundary of AdS_3 is located in this region where the theory can be described safely in terms of these free fields.

The quantum theory is constructed similarly as Liouville theory where the interaction term gets renormalized as

$$\mathcal{S}_+ = \int d^2z \, \beta\bar{\beta}e^{-\frac{2}{\alpha_+}\phi} \,. \tag{2.9}$$

This term in the action can be interpreted as a screening charge operator in correlation functions when the theory is considered as a Coulomb gas model. In this case there is however another screening operator which is equally suitable for the purpose of guaranteeing the charge conservation condition [5, 13] (see section 2.2 below), namely

$$\mathcal{S}_- = \int d^2z \, (\beta\bar{\beta})^{\frac{\alpha_+^2}{2}} e^{-\alpha_+\phi} \,. \tag{2.10}$$

These two perturbations (2.9) and (2.10) were discussed in reference [14] in the framework of the AdS/CFT correspondence, where it was pointed out that, unlike $\mathcal{S}_+$, $\mathcal{S}_-$ cannot reconstruct the AdS_3 geometry upon integrating out the auxiliary $\beta, \bar{\beta}$ fields. The situation is again similar to Liouville theory where two interaction terms seem necessary in order to get the correct pole structure of the correlation functions [15]. We shall discuss the contribution of both perturbations in the computation of the two and three point functions in section 2.2 below. Here let us notice that $\mathcal{S}_+$ can be considered as a small perturbation when $k \to 2$ for arbitrary ϕ whereas it is finite for $k \to \infty$. The exponential term in $\mathcal{S}_-$ instead is small when $k \to \infty$ but it tends to one for $k \to 2$.

Vertex operators creating physical states are needed in order to compute correlation functions. In the next subsection we review the properties of the operators that have been proposed in the literature.

2.1 Primary fields and vertex operators

The primary fields of the SL(2) conformal theory $\Phi^j_m(z)$ satisfy the following OPEs with the currents

$$J^+(z)\Phi^j_m(w) = \frac{(j-m)}{z-w}\Phi^j_{m+1}(w) + \cdots$$

$$J^3(z)\Phi^j_m(w) = \frac{m}{z-w}\Phi^j_m(w) + \cdots$$
$$J^-(z)\Phi^j_m(w) = \frac{(-j-m)}{z-w}\Phi^j_{m-1}(w) + \cdots . \tag{2.11}$$

They can be associated to differentiable functions on H_3^+, which can be decomposed in terms of representations of $\mathrm{SL}(2)\times\overline{\mathrm{SL}(2)}$. The differentiable operatos are the zero modes of the algebra. A convenient plane wave normalizable basis for $\mathcal{L}^2(H_3^+)$ is given by [21, 8]

$$\Phi_j(z,x) = \frac{2j+1}{\pi}\left(|\gamma - x|^2 e^{\phi/\alpha_+} + e^{-\phi/\alpha_+}\right)^{-2j-2}, \tag{2.12}$$

where $(x,\bar{x})$ are auxiliary complex variables. Spin j and $-j-1$ representations are equivalent and consequently the functions $\Phi_j(z,x)$ and $\Phi_{-j-1}(z,x)$ satisfy the following relation

$$\Phi_j(z,x) = \frac{R(j)}{\pi}\int d^2y\, |x-y|^{-4j-4}\Phi_{-j-1}(z,y)\,, \tag{2.13}$$

where $R(j)$ is the reflection coefficient verifying

$$R(j)R(-j-1) = -(2j+1)^2\,. \tag{2.14}$$

We are interested in the near boundary limit, i.e. $\phi\to\infty$. The expansion of Φ_j around $\phi\approx\infty$ was worked out in detail in references [22] where it was remarked that the behavior of the functions (2.12) changes as one approaches $j=-1/2$. In fact, as $\phi\to\infty$,

$$\lim_{\phi\to\infty}\Phi_j(x,\bar{x}) = e^{\frac{2j}{\alpha_+}\phi}\delta^{(2)}(\gamma - x) + \frac{2j+1}{\pi}|\gamma - x|^{-4j-4}e^{\frac{-2j-2}{\alpha_+}\phi} + \mathcal{O}\left(e^{\frac{1}{\alpha_+}j\phi}\right), \tag{2.15}$$

and it is easy to see that the first term is leading for $j>-1/2$ whereas the second one dominates for $j<-1/2$.

Fourier transforming the leading terms in this expansion as

$$\mathcal{V}_{j,m,\bar{m}}(z,\bar{z}) = \int d^2x\, \Phi_j(x,\bar{x})x^{j-m}\bar{x}^{j-\bar{m}}\,, \tag{2.16}$$

one obtains

$$\mathcal{V}_{j,m,\bar{m}} = V_{j,m,\bar{m}} + \frac{(2j+1)}{\pi}\frac{\Gamma(j-m+1)\Gamma(j+\bar{m}+1)\Gamma(-2j-1)}{\Gamma(-j+\bar{m})\Gamma(-j-m)\Gamma(2j+2)}V_{-1-j,m,\bar{m}}\,, \tag{2.17}$$

where

$$V_{j,m,\bar{m}} = \gamma^{j-m}\bar{\gamma}^{j-\bar{m}}e^{\frac{2j}{\alpha_+}\phi}\,. \tag{2.18}$$

Notice that equation (2.16) is well defined if $m-\bar{m}\in\mathbb{Z}$. Let us remark that, except for a j,m dependent factor, the field dependence of both terms in expression (2.17) is related by $j\leftrightarrow -j-1$.

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Consider now the relation (2.13). Fourier transforming both sides one finds

$$\mathcal{V}_{j,m,\bar{m}} = R(j)\frac{\Gamma(j-m+1)\Gamma(j+\bar{m}+1)\Gamma(-2j-1)}{\Gamma(-j+\bar{m})\Gamma(-j-m)\Gamma(2j+2)}\mathcal{V}_{-j-1,m,\bar{m}}\,, \tag{2.19}$$

and consequently the relative weight between both terms in the vertex operator has to be modified as

$$\mathcal{R}(j,m) = R(j)\frac{\Gamma(j-m+1)\Gamma(j+\bar{m}+1)\Gamma(-2j-1)}{\Gamma(-j+\bar{m})\Gamma(-j-m)\Gamma(2j+2)}\,. \tag{2.20}$$

The classical factor $\frac{2j+1}{\pi}$ has been replaced by the reflection coefficient $R(j)$ which was found in reference [8, 7] to be

$$R(j) = \frac{(2j+1)}{\pi}\left(\frac{\Gamma(1-\rho)}{\Gamma(1+\rho)}\right)^{2j+1}\frac{\Gamma(1+(2j+1)\rho)}{\Gamma\left(1-(2j+1)\rho\right)}, \tag{2.21}$$

where $\rho = -(k-2)^{-1}$. Observe that it reduces to $(2j+1)/\pi$ as $k\to\infty$.

Therefore the final form of the vertex operator in the large ϕ limit is

$$\mathcal{V}_{j,m,\bar{m}} = V_{j,m,\bar{m}} + \mathcal{R}(j,m)V_{-j-1,m,\bar{m}}\,. \tag{2.22}$$

The first term above reproduces the standard form of the vertex operator used in the Wakimoto representation of the theory [10, 11]. This term is dominant in the large ϕ limit for states in the discrete representation satisfying the unitarity bound $-1/2 < j < (k-3)/2$. However notice that both terms contribute to the same order for states in the continuous representation, i.e. $j = -1/2 \pm i\lambda$, $\lambda \in \mathbb{R}$, $m \in \mathbb{R}$, and thus the full expression should be used in this case.

For $j=-1/2$ there is a "resonance" and the leading term in the expansion is

$$\lim_{\phi\to\infty}\Phi_{-1/2} \sim \frac{\phi}{\alpha_+}\delta(\gamma-x)\delta(\bar{\gamma}-\bar{x})e^{-\phi/\alpha_+}\,. \tag{2.23}$$

Observe that the Fourier transform of this expression belongs to the following functional form

$$\hat{V}_{j,m,\bar{m}} = \frac{\phi}{\alpha_+}\gamma^{j-m}\bar{\gamma}^{j-\bar{m}}e^{\frac{2j}{\alpha_+}\phi} \tag{2.24}$$

evaluated at $j=-\frac{1}{2}$. The operators (2.24) have logarithmic structure in the Virasoro algebra, namely

$$L_0\hat{V}_{j,m} = -\frac{j(j+1)}{k-2}\hat{V}_{j,m} - \frac{1+2j}{k-2}V_{j,m}\,. \tag{2.25}$$

However they are indeed primary fields for the particular case $j=-1/2$. These objects are called prelogarithmic or puncture operators and they form a Jordan block of the $\mathrm{SL}(2)_k$ Kac-Moody algebra.

2.2 Two and three point functions

In this section we review the computation of two and three point tachyon amplitudes performed in the Wakimoto representation of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ black hole background in reference [11]. We follow closely the steps performed by K. and M. Becker but we are more general, i.e. we consider the two interaction terms (2.9) and (2.10) in the action.

Let us start by recalling the calculation of correlation functions of tachyon vertex operators, namely

$$\mathcal{A}^{j_1,j_2,\ldots,j_N}_{m_1,m_2,\ldots,m_N}(z_1,z_2,\ldots,z_N) = \int \prod_{i=1}^{N} d^2z_i \left\langle \prod_{i=1}^{N} V_{j_i,m_i}(z_i) \right\rangle . \qquad (2.26)$$

The expectation value is taken with the action

$$\begin{aligned} S_{\mu,\tilde{\mu}}[\beta,\gamma,\phi] = {}& \frac{1}{4\pi}\int d^2z \left(\partial\phi\bar{\partial}\phi - \frac{2}{\alpha_+}R\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}\right) - \\ & - \frac{1}{4\pi}\int d^2z \left(\mu\beta\bar{\beta}e^{-\frac{2}{\alpha_+}\phi} + \tilde{\mu}\beta^{\frac{\alpha_+^2}{2}}\bar{\beta}^{\frac{\alpha_+^2}{2}}e^{-\alpha_+\phi}\right) \end{aligned} \qquad (2.27)$$

as discussed above, where two coupling constants μ and $\tilde{\mu}$ have been inserted. In the case of the 2D black hole, μ is related to the black hole mass (see [16]).

Consistently with the perturbative scheme adopted for the computation, the vertex operators considered correspond to the large ϕ limit of the primary fields for $j > -1/2$ and $m_i = \bar{m}_i$. The large ϕ limit of the vertex operators creating tachyon states in the coset theory are

$$V_{j,m,\bar{m}} =: \gamma^{j-m}\bar{\gamma}^{j-\bar{m}}e^{\frac{2j}{\alpha_+}\phi}e^{im\sqrt{\frac{2}{k}}X} : . \qquad (2.28)$$

The compact free boson X was introduced in [16] to gauge the U(1) subgroup. The integration over its zero mode yields the following conservation law

$$\sum_{i=1}^{N} m_i = 0\,. \qquad (2.29)$$

Other than that, the contribution of X completes the conformal properties of the N-point functions, therefore we omit explicit reference to this field in the correlators.

The functional integral over ϕ can be performed as usual splitting the field ϕ into zero mode and oscillator parts, $\phi(z) = \phi_0 + \tilde{\phi}(z)$. Using the Gauss-Bonnet theorem and performing some algebraic manipulations, we find on the sphere

$$\left\langle \prod_{n=1}^{N} V_{j_n,m_n,\bar{m}_n} \right\rangle_{\mu,\tilde{\mu}} = \Gamma\left(-s_-\right)\Gamma\left(-s_+\right)\mu^{s_+}\tilde{\mu}^{s_-}\times$$

$$\times \left\langle \prod_{n=1}^{N} V_{j_n,m_n,\bar{m}_n} \prod_{r=1}^{s_+} \mathcal{S}_+(w_r) \prod_{t=1}^{s_-} \mathcal{S}_-(y_t) \right\rangle_{\mu=0,\tilde{\mu}=0} \times$$
$$\times\, \delta\left(s_- \frac{\alpha_+^2}{2} + s_+ - 1 - \sum_{n=1}^{N} j_n \right), \tag{2.30}$$

where we have absorbed an overall constant in the definition of the path integral. This expression shows that the interaction terms play the role of screening operators

$$\mathcal{S}_+ = \int d^2y\, \beta(y)\bar{\beta}(\bar{y}) e^{-\frac{2}{\alpha_+}\phi(y,\bar{y})} \tag{2.31}$$

and

$$\mathcal{S}_- = \int d^2w\, \beta(w)^{\alpha_+^2/2} \bar{\beta}(\bar{w})^{\alpha_+^2/2} e^{-\alpha_+\phi(w,\bar{w})}, \tag{2.32}$$

which have to be introduced in order to satisfy the conservation law

$$(k-2)s_- + s_+ = \sum_{i=1}^{N} j_i + 1, \tag{2.33}$$

arising from the integration over ϕ_0. Notice that s_- and s_+ are in general non integer, and they can be complex numbers if the states belong to the continuous representation. Without loss of generality we shall work out in full detail the case $s_+ = 0$, contrary to the choice $s_- = 0$ performed in reference [11]. At the end we shall compare the results obtained in both cases.

Let us start by considering two point functions of tachyon vertex operators. Conformal invariance allows one to determine the general structure of the two point functions, namely

$$\langle V_{j_1,m_1}(z_1) V_{j_2,m_2}(z_2)\rangle = |z_1 - z_2|^{-4\Delta_1} \left[A(j_1)\delta(j_1 + j_2 + 1) + B(j_1)\delta(j_1 - j_2) \right], \tag{2.34}$$

where $\Delta_1 = j_1(j_1+1)\rho + m_1^2/k$ is the conformal weight of the vertex operators (2.28) in the coset theory.

The correlators to be computed in order to determine $A(j)$ and $B(j)$ are

$$\int \prod_{i=1}^{s_-} d^2w_i \left\langle \gamma_{(z_1)}^{j_1-m_1} \gamma_{(z_2)}^{j_2-m_2} \prod_{i=1}^{s_-} \beta_{(w_i)}^{k-2} \right\rangle \times \text{c.c.} \times \left\langle e^{\frac{2j_1}{\alpha_+}\phi(z_1,\bar{z}_1)} e^{\frac{2j_2}{\alpha_+}\phi(z_2,\bar{z}_2)} \prod_{i=1}^{s_-} e^{-\alpha_+\phi(w_i,\bar{w}_i)} \right\rangle, \tag{2.35}$$

where we have renamed $\tilde{\phi} = \phi$, and the conservation law (2.33) to be considered in this case is

$$s_- = -\rho(j_1 + j_2 + 1). \tag{2.36}$$

The $\beta - \gamma$ correlator can be computed by bosonization, introducing as usual two ordinary bosons u and v such that

$$\beta = -i\partial v e^{iv-u}, \qquad \gamma = e^{u-iv}, \tag{2.37}$$

where

$$\langle u(z)u(w)\rangle = \langle v(z)v(w)\rangle = -\ln(z-w)\,. \tag{2.38}$$

This allows to make sense of non positive integer powers of γ, whereas non integer numbers of β fields are not well defined. Technical difficulties arising from the occurrence of fractional powers of $\beta-\gamma$ fields have been dealt with in references [17]. However we shall proceed as if there were an integer number of screening operators with an integer power of β fields and at the end the condition (2.36) for s_- will be imposed, assuming the expressions are defined by analytic continuation in s_-. The agreement of the results with those obtained by other approaches supplies the justification for this procedure.

Clearly the computation of the first term $A(j)$ in (2.34) requires no screening operators. Contractions from the γ's of the vertex operators are equal to one and the contribution from the exponentials reproduce the overall factor $|z_1-z_2|^{-4\Delta_1}$. Therefore, $A(j)=1$.

The term $B(j)$ in (2.34) can be computed similarly as in reference [11]. It is convenient to fix the tachyon vertices at $z_1=0$ and $z_2=1$ and the position of one screening operator at $w_{s_-}=\infty$ in order to factor out the $\mathrm{SL}(2,\mathbb{C})$ invariant volume.

The contribution of the $\beta-\gamma$ system can be obtained generalizing the procedure in reference [11], where it was found that

$$\left\langle \gamma_{(0)}^{j_1-m_1}\gamma_{(1)}^{j_2-m_2}\prod_{i=1}^{s}\beta_{(w_i)}\right\rangle = \mathcal{P}^{-1}\frac{\partial^s\mathcal{P}}{\partial w_1\cdots\partial w_s}\,, \tag{2.39}$$

with

$$\mathcal{P}=\prod_{i=1}^{s} w_i^{m_1-j_1}(1-w_i)^{m_2-j_2}\prod_{i<j}(w_i-w_j)\,. \tag{2.40}$$

In order to compute the correlator in equation (2.35) it is convenient to point split the insertion points of the screening operators, i.e. take $(k-2)s_-$ different points as $w_r^{(n)}, r=1,\ldots,s; n=1,\ldots,k-2$ and at the end take the limit where $w_r^{(n)}\to w_r$, $\forall n$. Thus we obtain

$$\left\langle \gamma_{(0)}^{j_1-m_1}\gamma_{(1)}^{j_2-m_2}\prod_{i=1}^{s_-}\beta_{(w_i)}^{k-2}\right\rangle = \lim_{w_i^{(n)}\to w_i^{(1)}=w_i}\mathcal{P}^{-1}\frac{\partial^{(k-2)s_-}\mathcal{P}}{\partial w_1^{(1)}\cdots\partial w_1^{(k-2)}\cdots\partial w_{s_-}^{(1)}\cdots\partial w_{s_-}^{(k-2)}}\,, \tag{2.41}$$

with

$$\mathcal{P}=\prod_{i=1}^{s_-}\prod_{n=1}^{k-2}(w_i^{(n)})^{m_1-j_1}(1-w_i^{(n)})^{m_2-j_2}\prod_{i<j}(w_i^{(n)}-w_j^{(m)})\,. \tag{2.42}$$

Therefore the contribution from the $\beta-\gamma$ correlator is

$$\left|\left\langle \gamma_{(0)}^{j_1-m_1}\gamma_{(1)}^{j_2-m_2}\prod_{i=1}^{s_-}\beta_{(w_i)}^{k-2}\right\rangle\right|^2 = (-)^{-\tilde\rho s_-}\triangle(1+j_1-m_1)\triangle(1+j_2-m_2)\prod_{i=1}^{s_-}|w_i|^{2\tilde\rho}|1-w_i|^{2\tilde\rho}\,, \tag{2.43}$$

where $\Delta(x)=\Gamma(x)/\Gamma(1-x)$, $\tilde\rho=\rho^{-1}=-(k-2)$ and $m_i=\bar m_i$.

Performing the ϕ contractions, the $(s_- - 1)$ integrals from the screening operators are

$$\int \prod_{i=1}^{s_- -1} d^2 w_i |w_i|^{4j_1+2\tilde{\rho}} |1-w_i|^{4j_2+2\tilde{\rho}} \prod_{i<j} |w_i - w_j|^{4\tilde{\rho}} \,. \tag{2.44}$$

These integrals have been computed by Dotsenko and Fateev [24] who found

$$\int \prod_{i=1}^{s} d^2 w_i \prod_{i=1}^{s} |w_i|^{2\alpha} |1-w_i|^{2\beta} \prod_{i<j}^{s} |w_i - w_j|^{4\sigma} = s! \pi^s \left(\triangle(1-\sigma)\right)^s \prod_{i=1}^{s} \triangle(i\sigma) \times$$
$$\times \prod_{i=0}^{s-1} \triangle(1+\alpha+i\sigma) \triangle(1+\beta+i\sigma) \triangle(-1-\alpha-\beta-(s-1+i)\sigma) \,. \tag{2.45}$$

Specifying the particular values of α and β in (2.44), the final result obtained for the term $B(j)$ in the two-point functions is

$$B(j) = (-\tilde{\mu}\pi\triangle(-\tilde{\rho}))^{s_-} \triangle(1+j-m)\triangle(1+j+m) s_- \tilde{\rho}^2 \triangle(1-s_-)\triangle(\tilde{\rho}s_-)\delta(m_1+m_2) \,, \tag{2.46}$$

where $j = j_1 = j_2$ and $m = m_1 = -m_2 = \bar{m}$.

Let us first compare this expression with the result obtained in reference [11], where only screening operators $\mathcal{S}_+$ were considered, namely[1]

$$B(j) = (-\pi\mu\triangle(-\rho))^{s_+} \triangle(1+j-m)\triangle(1+j+m) s_+ \rho^2 \triangle(1-s_+)\triangle(\rho s_+)\delta(m_1+m_2) \,, \tag{2.47}$$

where $s_+ = j_1 + j_2 + 1$.

The situation is similar to the case of the minimal models where there are two screening charges as well. The conformal properties of the correlation functions cannot be changed by the insertion of $\mathcal{S}_\pm$, as long as the correlators satisfy the charge balance. Therefore the results (2.46) and (2.47) should coincide, i.e. they should be independent of the screening operator used.

In order to see if this is the case let us recall that, as was shown in [11], the two-point functions can be obtained from the three-point functions containing one highest weight state V_{j_1,j_1}, taking the limit $j_1 = i\varepsilon \to 0$. We shall show below, after computing the three point functions, that in this case the term $B(j)$ is as found above (i.e. equations (2.46) and (2.47) from the direct calculation) with an extra factor $(s_-\tilde{\rho})^{-1}$ and $(s_+\rho)^{-1}$, respectively. Thus we shall compare the following expressions

$$B_-(j) = (-\pi\tilde{\mu}\triangle(-\tilde{\rho}))^{s_-} \triangle(1+j-m)\triangle(1+j+m)\tilde{\rho}\triangle(1-s_-)\triangle(\tilde{\rho}s_-)\delta(m_1+m_2) \,. \tag{2.48}$$

and

$$B_+(j) = (-\pi\mu\triangle(-\rho))^{s_+} \triangle(1+j-m)\triangle(1+j+m)\rho\triangle(1-s_+)\triangle(\rho s_+)\delta(m_1+m_2) \,. \tag{2.49}$$

[1] In comparison to [11] it should be noted that the result here contains an extra factor ρ^2

It is interesting to notice that these two expressions agree when replacing $s_+ = j_1 + j_2 + 1$ and $s_- = -\rho(j_1 + j_2 + 1)$, if the following expression holds

$$\pi\tilde{\mu}\triangle(-\tilde{\rho}) = (\pi\mu\triangle(-\rho))^{-\rho^{-1}} . \tag{2.50}$$

We shall find a similar relation in the computation of three point functions, after which we postpone some comments.

Let us now compare the results obtained above with others found in the literature. The expression for $B_+(j)$ (2.49) is exactly the Fourier transform of the result obtained in references [9, 7] (see [25]), namely

$$\langle \Phi_{j_1}(x,\bar{x})\Phi_{j_2}(x',\bar{x}')\rangle = \frac{k-2}{\pi}\left[\nu(k)\right]^{j_1+j_2+1}\frac{\Gamma(1-\frac{j_1+j_2+1}{k-2})}{\Gamma(\frac{j_1+j_2+1}{k-2})}|x-x'|^{-2(j_1+j_2+2)} , \tag{2.51}$$

where

$$\nu(k) = \frac{1}{\pi}\frac{\Gamma(1-\rho)}{\Gamma(1+\rho)} , \tag{2.52}$$

except for an irrelevant factor $(\rho/\pi^2)^{s_+}$ (notice that when $s_+ = 0$ this factor is 1, thus this does not affect the term $A(j)$ in the 2-point functions). Moreover, the Fourier transform can be performed even when $j_1 + j_2 + 1 = 0$. In order to see that, recall that the Dirac delta function can be written as

$$\delta^{(2)}(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\frac{|x|^{2(\varepsilon-1)}}{\Gamma(\varepsilon)} .$$

Then defining $j_2 = -1 - j_1 - \varepsilon$ and taking the limit $\varepsilon \to 0$, the term $A(j)$ (i.e. the term which is proportional to $\delta(j_1 + j_2 + 1)$) is recovered, namely

$$\lim_{\varepsilon\to 0}\langle \Phi_{j_1}(x,\bar{x})\Phi_{-1-j_1-\varepsilon}(x',\bar{x}')\rangle = \delta^2(x-x') .$$

It is remarkable that the free field approximation reproduces so accurately the exact result. We shall now show that the same agreement is found for the three point functions.

The computation of the three tachyon amplitudes goes along the same steps. It turns out that the simplest way to do it is to start with one highest weight tachyon, for example take $j_2 = m_2$. It was shown in reference [11] that a general three tachyon amplitude can then be expressed as a function of this one, acting with the currents J^-.

It is convenient to fix the positions of the vertices at $(z_1, z_2, z_3) = (0, 1, \infty)$. The amplitude is then

$$\begin{aligned}\mathcal{A}^{j_1,j_2,j_3}_{m_1,m_2,m_3} &= \Gamma(-s_-)\int\prod_{i=1}^{s_-}d^2w_i\left\langle \gamma_{(0)}^{j_1-m_1}\gamma_{(\infty)}^{j_3-m_3}\prod_{i=1}^{s_-}\beta_{(w_i)}^{k-2}\right\rangle \times \text{c.c.} \times \\ &\times \left\langle e^{\frac{2j_1}{\alpha_+}\phi(0)}e^{\frac{2j_2}{\alpha_+}\phi(1)}e^{\frac{2j_3}{\alpha_+}\phi(\infty)}\prod_{i=1}^{s_-}e^{-\alpha_+\phi(w_i,\bar{w}_i)}\right\rangle\end{aligned} \tag{2.53}$$

and the conservation laws are in this case

$$(k-2)s_- = j_1 + j_2 + j_3 + 1\,, \qquad m_1 + m_2 + m_3 = 0 \tag{2.54}$$

(notice that we are again using screening operators $\mathcal{S}_-$, unlike reference [11]).

The $\beta - \gamma$ correlator can be evaluated as in equation (2.41) above, with

$$\mathcal{P} = \prod_{i=1}^{s_-}\prod_{n=1}^{k-2}(w_i^{(n)})^{m_1-j_1}\prod_{i<j}(w_i^{(n)} - w_j^{(m)})\,, \tag{2.55}$$

and the result is now

$$\left|\left\langle \gamma_{(0)}^{j_1-m_1}\gamma_{(\infty)}^{j_3-m_3}\prod_{i=1}^{s_-}\beta_{(w_i)}^{k-2}\right\rangle\right|^2 = (-)^{-\tilde{\rho}s_-}\triangle(1+j_1-m_1)\triangle(1+j_3-m_3)\prod_{i=1}^{s_-}|w_i|^{2\tilde{\rho}}\,. \tag{2.56}$$

After performing the contractions of the exponentials we obtain

$$\mathcal{A}^{j_1,j_2,j_3}_{m_1,j_2,m_3} = (-)^{-\tilde{\rho}s_-}\triangle(1+j_1-m_1)\triangle(1+j_3-m_3)\tilde{\mathcal{I}}(j_1,j_2,j_3,k)\,, \tag{2.57}$$

where

$$\tilde{\mathcal{I}}(j_1,j_2,j_3,k) = \tilde{\mu}^{s_-}\Gamma(-s_-)\int\prod_{i=1}^{s_-}d^2w_i|w_i|^{4j_1+2\tilde{\rho}}|1-w_i|^{4j_2}\prod_{i<j}|w_i-w_j|^{4\tilde{\rho}} \tag{2.58}$$

can be evaluated using the Dotsenko Fateev formula (2.45) and the special function $G(j)$ which satisfies the following relations

$$\begin{aligned} G(j) &= G(-1-j-(k-2))\,, \\ G(j+1) &= \triangle((1+j)\rho)G(j)\,, \\ G(j-k+2) &= (k-2)^{2j+1}\triangle(1+j)G(j)\,. \end{aligned} \tag{2.59}$$

The final result can be expressed as

$$\tilde{\mathcal{I}}(j_1,j_2,j_3,k) = \frac{1}{k-2}[\pi\tilde{\mu}\tilde{\rho}^4\triangle(-\tilde{\rho})]^{s_-}D(j_1,j_2,j_3)\,, \tag{2.60}$$

where

$$D(j_1,j_2,j_3) = \triangle\left(-1-\sum_i j_i\right)\triangle(1+2j_2)\triangle(j_1-j_3-j_2)\triangle(-j_1-j_2+j_3)C(j_1,j_2,j_3) \tag{2.61}$$

and

$$\begin{aligned} C(j_1,j_2,j_3) = {} & \frac{G(-2-\sum_i j_i)}{G(-1)}\frac{G(-j_1+j_2-j_3-1)}{G(-2j_1-1)}\times \\ & \times\frac{G(j_1-j_3-j_2-1)G(-1-j_1-j_2+j_3)}{G(-2j_3-1)G(-2j_2-1)}\,. \end{aligned} \tag{2.62}$$

This computation can be repeated using the screening operators $\mathcal{S}_+$, as it was done in reference [11], instead of $\mathcal{S}_-$. The Dotsenko Fateev integral is in this case

$$\mathcal{I}(j_1,j_2,j_3,k) = \mu^{s_+}\Gamma(-s_+)\int\prod_{i=1}^{s_+} d^2w_i|w_i|^{-2-4\rho j_1}|1-w_i|^{-4\rho j_2}\prod_{i<j}|w_i-w_j|^{4\rho}\,. \quad (2.63)$$

Taking into account the multiplicity factor from the $\beta-\gamma$ system, the final result is

$$\mathcal{A}^{j_1,j_2,j_3}_{m_1,j_2,m_3} = (-)^{-s_+}\triangle(1+j_1-m_1)\triangle(1+j_3-m_3)\mathcal{I}(j_1,j_2,j_3,k)\,, \quad (2.64)$$

where

$$\mathcal{I}(j_1,j_2,j_3,k) = \frac{1}{k-2}[\pi\mu\rho^4\triangle(-\rho)]^{s_+}D(j_1,j_2,j_3)\,. \quad (2.65)$$

Therefore here again, similarly as in the computation of the two point functions, both results (2.57) and (2.64) are related by an exchange of $\rho\leftrightarrow\tilde{\rho}$ and $s_+\leftrightarrow s_-$.

It is easy to see that (2.64) completely agrees with the Fourier transform of the result obtained in reference [8] (except for the factor ρ^{4s_+}), namely

$$\int d^2x_1d^2x_2d^2x_3|x_1|^{2(j_1-m_1)}|x_2|^{2(j_2-m_2)}|x_3|^{2(j_3-m_3)}|x_1-x_2|^{-2(j_1+j_2-j_3)}\times$$
$$\times|x_1-x_3|^{-2(j_1+j_3-j_2)}|x_2-x_3|^{-2(j_2+j_3-j_1)}C(j_1,j_2,j_3)\,, \quad (2.66)$$

which can be explicitly performed in the case $j_2=m_2$. The pole structure of this expression was analysed in reference [25].

Let us now make some comments about equation (2.50). A similar relation is discussed in reference [15] in connection with Liouville theory, where it reflects the self duality of the theory when $\rho\leftrightarrow\tilde{\rho}$ and moreover it seems necessary to produce the correct pole structure of the correlators. However the SL(2)/ U(1) coset theory is not obviously self dual. Indeed it has been conjectured to be equivalent to the Sine-Liouville model, i.e. $c=1$ CFT coupled to a Liouville field [18, 19] (see [20] for the fermionic generalization of this duality). In this case there is a strong/weak coupling duality on the world sheet. The cigar CFT becomes weakly coupled in the limit $k\to\infty$ whereas it is strongly coupled in the limit $k\to 2$, where the Sine-Liouville theory becomes weakly coupled. Recalling the comment below equations (2.9) and (2.10), the screening operators can be observed to satisfy a similar relation, i.e. $\mathcal{S}_-$ is weakly coupled when $k\to\infty$ and strongly coupled when $k\to 2$, contrary to $\mathcal{S}_+$. Furthermore both perturbations satisfy a relation of the same sort as the corresponding interaction terms in Liouville theory, namely $\mathcal{S}_-=(\mathcal{S}_+)^{-1/\rho}$.

It is not clear to us what conclusion can be drawn from these observations. Expressions (2.48) and (2.49) might describe the two point functions in two different regimes of the same theory (similarly (2.57) and (2.64) for the three point functions). Since the calculations are perturbative one would expect that (2.48) and (2.57) provide the correct answer when $k\to\infty$ and (2.49) and (2.64) when $k\to 2$. However if the identity (2.50) is taken seriouly it might be indicating a hidden self duality of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset theory.

For the sake of completeness we end this section with a derivation of equation (2.49), i.e. the two-point function obtained from the three point function containing one highest weight state, $j_2 = m_2$ in the limit $j_2 = i\varepsilon \to 0$. Considering screening operators $\mathcal{S}_+$ we obtain

$$\mathcal{A}^{j_1,i\varepsilon,j_3}_{m_1,i\varepsilon,m_3} = (-)^{s_+}\triangle(1+j_1-m_1)\triangle(1+j_3-m_3)I(j_1,j_3,k)\,, \tag{2.67}$$

where $I(j_1,j_3,k)$ is given by (2.63) with $j_2 = i\varepsilon$, i.e. simplifying the products of $\triangle$-functions we obtain

$$\begin{aligned} I(j_1,j_3,k) = (\pi\mu\triangle(-\rho))^{s_+}\triangle(1-s_+)\triangle(\rho s_+)\times \\ \times \lim_{\varepsilon\to 0}\triangle(1-2\rho\varepsilon i)\triangle((\varepsilon i - j_1 + j_3)\rho)\triangle((\varepsilon i + j_1 - j_3)\rho)\,. \end{aligned} \tag{2.68}$$

The limit $\varepsilon \to 0$ can be evaluated using that in this region

$$\Gamma(-n+\varepsilon) = \frac{(-)^n}{\varepsilon\Gamma(n+1)} + \mathcal{O}(1) \quad \text{for} \quad n \in \mathbb{N}\,, \tag{2.69}$$

and taking into account the following representation of the delta function

$$\delta(j_1 - j_3) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2 + (j_1-j_3)^2}\,. \tag{2.70}$$

Putting all this together we obtain $(2\pi i)$ times equation (2.49).

3. The spectral flow and new representations

Until now we have been considering the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ WZW model. In this section we extend the computations of the previous one to string theory in AdS_3. As it was pointed out in references [2, 3], the algebra (2.3) has a spectral flow symmetry given by

$$\begin{aligned} J^3_n \to \tilde{J}^3_n &= J^3_n - \frac{k}{2}\omega\delta_{n,0} \\ J^\pm_n \to \tilde{J}^\pm_n &= J^\pm_{n\pm\omega} \end{aligned} \tag{3.1}$$

and thus,

$$L_n \to \tilde{L}_n = L_n + \omega J^3_n - \frac{k}{4}\omega^2\delta_{n,0}\,, \tag{3.2}$$

where $\omega \in \mathbb{Z}$ is the winding number. The spectral flow generates new representations of the $\mathrm{SL}(2,R)$ algebra. The Hilbert space of string theory in AdS_3 can be consequently extended $\mathcal{H} \to \mathcal{H}_\omega$ in order to include the states $|\tilde{j},\tilde{m},\omega\rangle$ obtained by spectral flow, which satisfy the following on-shell condition

$$(L_0 - 1)\,|\tilde{j},\tilde{m},\omega\rangle = \left(-\frac{\tilde{j}(\tilde{j}+1)}{(k-2)} - \omega\tilde{m} - \frac{k}{4}\omega^2 + \mathcal{N} - 1\right)|\tilde{j},\tilde{m},\omega\rangle = 0 \tag{3.3}$$

$\mathcal{N}$ being the excitation level of the string. The new representations are denoted by $\hat{\mathcal{D}}_{\tilde{j}}^{\pm,\omega}$ and $\hat{\mathcal{C}}_{\tilde{j}}^{\omega}$ and they consist of the spectral flow of the discrete (highest and lowest weight) and continuous series respectively. It was shown in reference [3] that the spectrum of the free theory is closed under the spectral flow symmetry if the spin $\tilde{j}$ of the physical states in the discrete representations is restricted to $\tilde{j} < \frac{k-3}{2}$.

The spectrum of string theory consists then of a product of left and right representations $\hat{\mathcal{C}}_{\tilde{j},L}^{\omega} \times \hat{\mathcal{C}}_{\tilde{j},R}^{\omega}$ and $\hat{\mathcal{D}}_{\tilde{j},L}^{\pm,\omega} \times \hat{\mathcal{D}}_{\tilde{j},R}^{\pm,\omega}$ with the same amount of spectral flow and the same spin $\tilde{j}$ on the holomorphic and antiholomorphic parts and with $-1/2 < \tilde{j} < (k-3)/2$. The partition function containing the spectral flow of the discrete representations with this bound on the spin $\tilde{j}$ was shown to be modular invariant in [3]. Moreover, the partition function for thermal AdS_3 backgrounds was also found to be modular invariant and consistent with this spectrum in [26]. From now on we drop the tilde on $\tilde{j}, \tilde{m}$.

In reference [4] we considered this theory as the coset model $\frac{H_3^+}{\mathrm{U}(1)} \times$time. A direct extension of Dotsenko's method to compute the conformal blocks in the compact SU(2) CFT to the non-compact H_3^+ group manifold was found adequate to deal with the spectral flow symmetry in vertex operators and scattering amplitudes. Two free scalar fields were introduced: $X(z)$ gauges the U(1) subgroup as before and the timelike scalar field $Y(z)$ bosonizes the J^3 current as

$$J^3(z) = -i\sqrt{\frac{k}{2}}\partial Y(z)\,. \tag{3.4}$$

Their propagators are $\langle Y(z)Y(w)\rangle = -\langle X(z)X(w)\rangle = \ln(z-w)$.

In terms of Wakimoto free fields the vertex operators in the unflowed sector of the theory can be written in the form

$$V_{j,m,\bar{m}} = \gamma^{j-m}\bar{\gamma}^{j-\bar{m}} e^{\frac{2j}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}mX} e^{i\sqrt{\frac{2}{k}}mY}\,. \tag{3.5}$$

Taking into account the spectral flow, for every field $V_{j,m}$ in the sector $\omega = 0$ one can write a field in the sector twisted by ω as

$$V_{j,m}^{\omega} = \gamma^{j-m}\bar{\gamma}^{j-\bar{m}} e^{\frac{2j}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}mX} e^{i\sqrt{\frac{2}{k}}(m+\omega k/2)Y}\,. \tag{3.6}$$

This vertex operator has the following conformal weight

$$\Delta(V_{j,m}^{\omega}) = -\frac{j(j+1)}{k-2} - m\omega - \frac{k\omega^2}{4} \tag{3.7}$$

(see reference [27] for an alternative approach to the description of winding strings).

The N-point functions were constructed in [4] (the reader is referred to that reference for details of the construction). They take the form

$$\mathcal{A}_N^{0,\pm} = \left\langle \prod_{i=1}^{N-1} V_{j_i,m_i}^{\omega_i}(z_i)\tilde{V}_{j_N,j_N}^{\omega_N(0),(\pm)}(z_N)\prod_{n=1}^{s}\mathcal{S}(u_n)\right\rangle_{0,\pm}\,, \tag{3.8}$$

where the conjugate highest weight operators $\tilde{V}^{\omega(0),(\pm)}_{j,j}$ are needed to avoid redundant integrations. Two of these conjugate vertices were found to be

$$\tilde{V}^{\omega(0)}_{j,j} = \beta^{2j+k-1}\bar{\beta}^{2j+k-1}e^{-\frac{2(j-1+k)}{\alpha_+}\phi}e^{i\sqrt{\frac{2}{k}}jX}e^{i\sqrt{\frac{2}{k}}(j+\frac{k}{2}\omega)Y} \tag{3.9}$$

and

$$\tilde{V}^{\omega(-)}_{j,j} = \beta^{2j}\bar{\beta}^{2j}e^{-\frac{(2j+k)}{\alpha_+}\phi}e^{i\sqrt{\frac{2}{k}}(j-\frac{k}{2})X}e^{i\sqrt{\frac{2}{k}}(j+\frac{k}{2}\omega)Y}\,. \tag{3.10}$$

The screening operators $\mathcal{S}$ in (3.8) can be either $\mathcal{S}_+$ or $\mathcal{S}_-$ or combinations of them, as discussed in section 2. Similarly to the SU(2) case the conjugate vertices have to be included in the conformal blocks in the Coulomb gas formalism to avoid redundant contour integrations. We now briefly review the procedure to be followed in order to find them.

Non-vanishing correlators must satisfy the charge asymmetry conditions which are determined by the operator conjugate to the identity. This is an operator that commutes with the currents and has zero conformal dimension. Three such operators were found in [4]

$$\tilde{\mathcal{I}}_0(z) = \beta^{k-1}e^{\frac{2(1-k)}{\alpha_+}\phi}\,, \qquad \tilde{\mathcal{I}}_+ = \gamma^{-k}e^{-\frac{k}{\alpha_+}\phi}e^{i\sqrt{\frac{k}{2}}X}\,, \qquad \tilde{\mathcal{I}}_- = e^{-\frac{k}{\alpha_+}\phi}e^{-i\sqrt{\frac{k}{2}}X} \tag{3.11}$$

They lead respectively to the following charge asymmetry conditions

$$\mathcal{C}^{(0)}: \qquad N_\beta - N_\gamma = k-1\,, \qquad \sum_i \alpha_i = \frac{2-2k}{\alpha_+}\,, \qquad \sum_i \xi_i = 0\,, \tag{3.12}$$

$$\mathcal{C}^{(+)}: \qquad N_\beta - N_\gamma = k\,, \qquad \sum_i \alpha_i = -\frac{k}{\alpha_+}\,, \qquad \sqrt{\frac{2}{k}}\sum_i \xi_i = \sqrt{\frac{k}{2}}\,, \tag{3.13}$$

and

$$\mathcal{C}^{(-)}: \quad N_\beta - N_\gamma = 0\,, \qquad \sum_i \alpha_i = -\frac{k}{\alpha_+}\,, \qquad \sqrt{\frac{2}{k}}\sum_i \xi_i = -\sqrt{\frac{k}{2}}\,, \tag{3.14}$$

where N_β, N_γ refer to the number of β, γ fields, α_i denotes the coefficient of the field $\phi(z_i)$ in the exponentials and ξ_i denotes the "*charge*" of the field $X(z_i)$. These have to be supplemented with an additional charge conservation law arising from exponentials of the field $Y(z)$, namely

$$\sum_i \Omega_i = \sum_i \left(m_i + \frac{\omega_i k}{2}\right) = 0\,, \tag{3.15}$$

where Ω_i denotes the "*charge*" of the field $Y(z_i)$. This is the energy conservation condition.

The conjugate representations for the highest weight operators (3.9) and (3.10) are found by asking that the two point functions $\langle V_{j,j}\tilde{V}_{j,-j}\rangle$ do not require screening operators to satisfy the charge asymmetry conditions $\mathcal{C}^{(0)}$ and $\mathcal{C}^{(-)}$ respectively, and it is easy to see that the conjugate operators $\tilde{V}^{\omega(+)}_{j,j}$ with respect to $\mathcal{C}^{(+)}$ do not have such a simple form.

Notice that a highest weight state is included in the definition of the correlation functions (3.8). This is done for the sake of simplicity. More generally other vertex operators can be defined by acting on the highest weight ones with the currents J^-, although in practice more complicated expressions are generated in this way. Hence the N-point functions to be considered in the coset theory take the form

$$\mathcal{A}^{0,-}_N = \left\langle \prod_{i=1}^{N-1} V^{\omega_i}_{j_i,m_i}(z_i)\tilde{V}^{\omega_N(0),(-)}_{j_N,j_N}(z_N)\prod_{n=1}^{s_+}\mathcal{S}_+(u_n)\prod_{m=1}^{s_-}\mathcal{S}_-(v_m)\right\rangle_{0,-} , \tag{3.16}$$

where the number of screening operators s_+, s_-, must satisfy the charge asymmetry conditions (3.12) or (3.14) respectively, which are determined by the conjugate vacuum state, and non-vanishing results require in addition the conservation law (3.15).

Let us stress that it is possible to construct correlators violating winding number conservation by, for instance, inserting conjugate operators $\tilde{V}^{\omega(-)}_{j,j}$ instead of direct ones into $\mathcal{A}^{(0)(-)}_N$. In fact, correlation functions containing K of these conjugate operators lead to $\sum_i \omega_i = -K$ when combining (3.12) or (3.14) with (3.15), whereas processes conserving winding number ($\sum_i \omega_i = 0$) are obtained when inserting direct vertex operators. Recall that it is possible to consider correlators containing up to $N-2$ conjugate operators of a different kind as the one which is required by the conjugate vacuum state, and thus the winding number conservation can be violated up to $N-2$ units (this possibility was proposed in [18, 25]).

Let us now proceed to compute two and three point functions using this formalism.

The general structure of the two point functions is determined from conformal invariance to be as equation (2.34) above, namely

$$\left\langle V^{\omega_1}_{j_1,m_1}(z)V^{\omega_2}_{j_2,m_2}(w)\right\rangle = |z-w|^{-4\Delta_1}[A^{\omega}(j_1)\delta(j_1-j_2)+B^{\omega}(j_1)\delta(j_1+j_2+1)]\delta(m_1+m_2) . \tag{3.17}$$

Now the conformal dimension is given by $\Delta_1 = j_1(j_1+1)\rho - m_1\omega_1 - k\omega_1^2/4$. The terms $A^{\omega}(j)$ and $B^{\omega}(j)$ can be computed as

$$\begin{aligned}\langle V^{\omega_1}_{j_1,m_1}\tilde{V}^{\omega_2(-)}_{j_2,j_2}\rangle_- = & \int\prod_{i=1}^{s_+-1} d^2w_i \left\langle \gamma_{(0)}^{j_1-m_1}\beta_{(1)}^{2j_2}\prod_{i=1}^{s_+-1}\beta_{(w_i)}\right\rangle \times \text{c.c.} \times \\ & \times \left\langle e^{\frac{2j_1}{\alpha_+}\phi(0)}e^{-\frac{(2j_2+k)}{\alpha_+}\phi(1)}\prod_{i=1}^{s_+-1}e^{-\frac{2}{\alpha_+}\phi(w_i,\bar{w}_i)}\right\rangle , \end{aligned} \tag{3.18}$$

where we have taken the conjugate vertex with respect to conditions (3.14), but it is easy to repeat the calculation for $\tilde{V}^{\omega(0)}_{j,m}$ in (3.9) (using the conditions (3.12)) and the result is identical. We are omittting the X and Y exponentials since other than completing the conformal weight and determining the conservation laws, the result does not depend on these contributions.

For the sake of simplicity we are using the screening operator $\mathcal{S}_+$. The conservation laws $\mathcal{C}^{(-)}$ take now the form

$$s_+ = j_1 - j_2\,, \qquad m_1 + j_2 = 0\,, \qquad \omega_1 + \omega_2 = 0 \tag{3.19}$$

(again we take $m_i = \bar{m}_i$).

Clearly it is now $A^\omega(j)$, i.e. the term proportional to $\delta(j_1 - j_2)$, the one that requires no screening operators. In this case the contribution of the $\beta - \gamma$ system is

$$\left\langle \gamma_{(z)}^{j-m_1} \beta_{(w)}^{2j} \right\rangle = \frac{\Gamma(2j+1)}{(z-w)^{2j}}\,, \tag{3.20}$$

and the correlators of the ϕ, X, Y exponentials reconstruct the conformal dimension of the two point function. So in order to normalize (3.17) as before, the conjugate highest weight vertex operator has to be defined as

$$\tilde{V}_{j,j}^{\omega(-)} = \frac{1}{\Gamma(2j+1)^2} \beta^{2j} \bar{\beta}^{2j} e^{-\frac{(2j+k)}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}(j-\frac{k}{2})X} e^{i\sqrt{\frac{2}{k}}(j-\frac{k}{2}\omega)Y} \tag{3.21}$$

and thus, $A^\omega(j) = 1$.

To compute the term $B^\omega(j)$, the $\beta - \gamma$ system can be treated as described in section 2.2, and the result is

$$\left\langle \gamma^{j_1-m_1}(0)\beta^{2j_2}(1)\prod_{i=1}^{s_+}\beta(w_i) \right\rangle = \frac{\Gamma(-j_1+m_1+2j_2+s_+)}{\Gamma(-j_1+m_1)} \prod_{i=1}^{s_+} w_i^{-1}\,. \tag{3.22}$$

Taking into account the antiholomorphic contribution and the ϕ correlator, the Dotsenko-Fateev integrals to be computed in this case are

$$\int \prod_{i=1}^{s_+-1} d^2w_i |w_i|^{-2+8j_1/\alpha_+^2} |1-w_i|^{-4(2j_2+k)/\alpha_+^2} \prod_{i<j}^{s_+-1} |w_i - w_j|^{-8/\alpha_+^2}\,. \tag{3.23}$$

The final result obtained is

$$\begin{aligned} B^\omega(j) = {} & \frac{\Gamma(j_2+m_1)^2}{\Gamma(1+2j_2)^2\Gamma(m_1-j_1)^2} (\pi\mu\triangle(-\rho))^{s_+} \times \\ & \times s_+\rho^2 \triangle(1-s_+)\triangle(\rho s_+)\delta(m_1+m_2)\delta(\omega_1+\omega_2)\,. \end{aligned} \tag{3.24}$$

It is easy to show that this expression agrees with the term $B(j)$ in equation (2.47) found in the section 2.2. In order to see this, one has to replace j_2 in the steps performed to arrive at (2.47) by $-1-j_2$ (as it is clearly needed to identify the terms

$B(j) \leftrightarrow B^{\omega}(j)$), and $m_1 = -j_2$, and the equality $\Gamma(0)/\Gamma(-s) = (-)^s\Gamma(s+1)$ has to be used. Then we can write the result more suggestively as

$$\begin{aligned} B^{\omega}(j) = & \triangle(1+j-m)\triangle(1+j+m)(-\pi\mu\triangle(-\rho))^{s_+} \times \\ & \times s_+\rho^2\triangle(1-s_+)\triangle(\rho s_+)\delta(m_1+m_2)\delta(\omega_1+\omega_2)\,. \end{aligned} \tag{3.25}$$

where $j = j_1 = -1 - j_2$, $m = m_1$.

Therefore the two point functions computed in reference [11] for the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ black hole background coincide with those obtained here for string theory in AdS_3 using a modified Coulomb gas formalism. This agreement is not surprising since the conformal properties completely determine the theory and even if the vertex operators have a different representation they correspond to the same state (notice that we are in fact comparing only the $\mathrm{SL}(2,\mathbb{R})$ part of the correlators). Nevertheless it should be observed that this agreement confirms the consistency of the formalism we have developed.

The computation of the two point functions can be repeated using screening operators $\mathcal{S}_-$ and the result (2.48) for B_- is also reproduced (again replacing $j_2 \leftrightarrow -1-j_2$). Precise agreement is also found with the results obtained in the previous section for the three point functions conserving winding number, for example $\langle V^{\omega_1}_{j_1,m_1} V^{\omega_2}_{j_2,m_2} \tilde{V}^{\omega_3(0)}_{j_3,j_3}\rangle_0$ or $\langle V^{\omega_1}_{j_1,m_1} V^{\omega_2}_{j_2,m_2} \tilde{V}^{\omega_3(-)}_{j_3,j_3}\rangle_-$, give (2.64) or (2.57) when the screening operators $\mathcal{S}_+$ or $\mathcal{S}_-$ respectively are considered. In this case one has to identify $j_3 \leftrightarrow -1-j_3$. Details of the calculations are not included because they are tedious and similar to those already described fully in the section 2.2. However it is interesting to remark again that our prescription yields results for the two and three point functions matching the known exact results.

The novelty comes about when computing three point functions violating winding number conservation, for example

$$\begin{aligned} &\left\langle V^{\omega_1}_{j_1,m_1}(z_1)\tilde{V}^{\omega_2(-)}_{j_2,j_2}(z_2)\tilde{V}^{\omega_3(-)}_{j_3,m_3}(z_3)\right\rangle_- = \\ &\qquad \int\prod_{i=1}^{s_+} d^2w_i \left|\left\langle \gamma^{j_1-m_1}_{(z_1)}\frac{\beta^{2j_2}_{(z_2)}}{\Gamma(1+2j_2)}\frac{\beta^{2j_3}_{(z_3)}}{\Gamma(1+2j_3)}\prod_{i=1}^{s_+}\beta_{(w_i)}\right\rangle\right|^2 \times \\ &\qquad \times\left\langle e^{\frac{2j_1}{\alpha_+}\phi(z_1,\bar{z}_1)}e^{-\frac{(2j_2+k)}{\alpha_+}\phi(z_2,\bar{z}_2)}e^{-\frac{(2j_3+k)}{\alpha_+}\phi(z_3,\bar{z}_3)}\prod_{i=1}^{s_+}e^{-\frac{2}{\alpha_+}\phi(w_i,\bar{w}_i)}\right\rangle\,. \end{aligned} \tag{3.26}$$

The exponentials of X and Y are not explicitly included since they just complete the conformal weight of the correlator, but we consider their contribution to the conservation laws which are now given by (see (3.14) and (3.15))

$$s_+ = j_1 - j_2 - j_3 - \frac{k}{2}\,, \qquad m_1 + j_2 - \frac{k}{2} + j_3 = 0\,, \qquad \omega_1+\omega_2+\omega_3 = -1\,. \tag{3.27}$$

In order to compute the correlators we fix the points as usual at $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$. The $\beta - \gamma$ correlator gives

$$\left\langle \gamma_{(0)}^{j_1-m_1} \beta_{(1)}^{2j_2} \beta_{(\infty)}^{2j_3} \prod_{i=1}^{s_+} \beta_{(w_i)} \right\rangle = \frac{\Gamma(2j_2+2j_3+s_+-j_1+m_1)}{\Gamma(-j_1+m_1)} \prod_{i=1}^{s_+} w_i^{-1}, \tag{3.28}$$

and the Dotsenko Fateev integral to be computed after performing the ϕ contractions is

$$\int \prod_{i=1}^{s_+} d^2 w_i |w_i|^{-2-4j_1\rho} |1-w_i|^{4\rho(j_2+k/2)} \prod_{i<j} |w_i - w_j|^{4\rho}. \tag{3.29}$$

Evaluating the integrals and simplifying the products of the $\triangle$-functions using (2.59), the result is

$$\begin{aligned} \mathcal{A}_{m_1,j_2,j_3}^{j_1,j_2,j_3}(\delta\omega=-1) = & \frac{\Gamma(j_2+j_3-k/2+m_1)^2}{\Gamma(-j_1+m_1)^2\Gamma(2j_2+1)^2\Gamma(2j_3+1)^2} \times \\ & \times \frac{1}{k-2} (\pi\mu\rho^4\triangle(-\rho))^{s_+} D^\omega(j_1,j_2,j_3,k) \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} D^\omega(j_1,j_2,j_3,k) &= D\left(j_1, -j_2-\frac{k}{2}, -1-j_3, k\right) \\ &= \frac{G(j_1-j_2-j_3-\frac{k}{2})}{G(-1)} \frac{G(-1-j_1-j_2-\frac{k}{2}-j_3)}{G(-2j_1-1)} \times \\ &\quad \times \frac{G(1+j_1-j_2+j_3-\frac{k}{2})}{G(1+2j_3)} \frac{G(-j_1-j_2-\frac{k}{2}+j_3)}{G(-2j_2-k)} \end{aligned} \tag{3.31}$$

Notice that the quotient of $\Gamma-$ functions coming from the multiplicity of the $\beta-\gamma$ correlator and the normalization of the vertex operators can be written in a similar fashion as the expressions obtained previouly using the equality $\Gamma(0)/\Gamma(-s) = (-)^s\Gamma(s+1)$. Then finally we obtain

$$\begin{aligned} \mathcal{A}_{m_1,j_2,j_3}^{j_1,j_2,j_3}(\delta\omega=-1) = & \triangle(1+j_1-m_1)\triangle(-2j_2)\triangle(-2j_3) \times \\ & \times \frac{1}{k-2} (\pi\mu\rho^4\triangle(-\rho))^{s_+} D^\omega(j_1,j_2,j_3,k). \end{aligned} \tag{3.32}$$

Recall that the vertex operators with quantum numbers (j_2, m_2) and (j_3, m_3) create conjugate highest weight states. Thus the identification $j_2 \to -1-j_2$ (similarly for j_3) is convenient in the $\triangle$-functions above in order to compare this expression with (2.64).

Similar results are obtained if the conjugate vacuum state $\tilde{V}_{j,j}^{\omega(0)}$ is considered instead of $\tilde{V}_{j,j}^{\omega(-)}$. Moreover the same expression is found if the screening operators $\mathcal{S}_-$ are taken, as long as one replaces $\rho \to \tilde{\rho}$ and $s_+ \to s_-$.

The three point function violating winding number by one unit, equation (3.32), or the equivalent one obtained replacing $\rho \to \tilde{\rho}$, presents poles given by the four $G(x)$ functions in the numerator. These poles are located at

$$
\begin{aligned}
j_1 - j_2 - j_3 - \frac{k}{2} &= n + m(k-2)\,, &\qquad 1 + j_1 - j_2 + j_3 - \frac{k}{2} &= n + m(k-2) \\
-1 - j_1 - j_2 - \frac{k}{2} - j_3 &= n + m(k-2)\,, &\qquad -j_1 - j_2 - \frac{k}{2} + j_3 &= n + m(k-2)
\end{aligned}
\tag{3.33}
$$

where $(n,m) \in \mathbb{Z}^2_{\geq 0}$ or $(n,m) \in \mathbb{Z}^2_{<0}$. Like in the case of winding conserving processes some of these poles are outside the unitarity bound $-\frac{1}{2} < j < \frac{k-3}{2}$. The remaining poles are of the same sort as those considered pathologies in reference [25].

4. Summary and conclusions

In this paper we reviewed the computation of two and three point tachyon amplitudes in the free field Coulomb gas approach to string theory in the background of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ black hole. Two screening operators were considered, $\mathcal{S}_-$ and $\mathcal{S}_+$. We showed that the results obtained using $\mathcal{S}_+$, originally performed in reference [11], completely agree with previous calculations performed by other methods. This is surprising because the Wakimoto representation is expected in principle to provide a good description of the theory when $\phi \to \infty$ (i.e. far from the tip of the cigar) where the string coupling goes to zero, but in fact these results show that this approximation encodes the information about the full theory, at least up to the three point functions. This observation was made before in reference [7] where the free field approach was carried out using a different formalism. In particular, the non-perturbative term $\Gamma(1 - \frac{2j+1}{k-2})$ appears in this approximation. This term, which is a finite k effect, was observed in references [18, 19] to give rise to the same poles that appear in the Sine-Liouville theory which was conjectured to be the S-dual of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset theory.

The outcome of the Coulomb gas calculation examined in section 2 is unexpected also because some expressions are highly formal. Indeed it is difficult to make sense of non-integer powers of β fields in the correlators, so we have proceeded as if there were a positive integer number. This is a non trivial step, eventually justified in the light of the results accomplished, and it indicates that the analytic continuation in s_+ is well defined.

An alternative representation of the correlators is obtained when considering the screening operators $\mathcal{S}_-$. We have shown that the results obtained in this framework match the previous ones if the following relation holds

$$
\pi\tilde{\mu}\triangle(-\tilde{\rho}) = (\pi\mu\triangle(-\rho))^{-\rho^{-1}}\,. \tag{4.1}
$$

An equivalent expression was recently found within the context of Liouville theory in reference [15], where it was interpreted as reflecting the self duality of the theory.

In that case, additional poles in the three point functions were found when inserting two interaction terms in the path integral that are related by $\rho \leftrightarrow \rho^{-1}$. Instead, the screening operators considered here, $\mathcal{S}_+$ and $\mathcal{S}_-$, satisfy another relation, namely $\mathcal{S}_- = (\mathcal{S}_+)^{-1/\rho}$, which is also verified by the Liouville perturbations, but unlike Liouville theory the $\mathrm{SL}(2)/\mathrm{U}(1)$ coset model is not obviously self dual. In fact, as it was mentioned above, this model was conjectured to be related by a strong/weak coupling duality symmetry to the Sine-Liouville model [18, 19]. Strong and weak coupling correspond in this context to the limits $k \to \infty$ and $k \to 2$ and viceversa. Notice that $\mathcal{S}_+$ and $\mathcal{S}_-$ respectively can be considered as small perturbations in these regimes.

Let us repeat that the conclusions to be drawn from these observations are not clear to us. At first sight it seems that each screening operator is adequate to work in a different curvature region (recall that k is related to the radius of curvature of spacetime). However, since the coset theory presents so many similarities with Liouville theory, self duality might not be a priori an exception.

One hint about the resolution of these issues could be given by the four point functions. Some of them were computed in the free field approximation in reference [7], and the results obtained were shown to solve the Knizhnik-Zamolodchikov equation. However computations are not yet available in the general case, where non trivial singularities are expected in the limit $z \to x$.

We also performed the computation of the two and three point functions in string theory in AdS_3. This theory was considered as the $\mathrm{SL}(2)/\mathrm{U}(1)$ coset times the state space of a timelike free boson. The extension of the formalism developed by Dotsenko [5] for SU(2) CFT proved to be adequate to explicitly introduce the spectral flow symmetry and winding number. Conjugate vertex operators in the correlators allowed to define scattering processes violating winding number conservation. The results obtained for the two and three point tachyon amplitudes conserving winding number exhibited exact coincidence with previous results. This was expected due to the conformal nature of the theory. However it is interesting to stress that this agreement also confirms the consistency of the prescription developed in reference [4].

Regarding this question it is interesting to notice that the conjugate operators have been defined for highest weight states. It is easy to generalize the procedure applied here to include vertex operators creating more general states in the discrete representation. This can be done by acting on the correlators with the currents J^- as was shown in reference [11]. However the results obtained apply also to states belonging to the continuous representation as it is indicated by the equivalence of the present results with correlators obtained by other approaches. Actually this is a relevant case in string theory, where one can define a notion of S-matrix for the long strings which are precisely in the spectral flow of the continuous representation. As it is described in reference [3] asymptotic states consisting of long strings can approach the center of AdS_3 and scatter back to the boundary. A non trivial result is the fact that in this process the winding number could in principle change. Our prescription allows to compute N-point functions violating winding number conservation for up to $N-2$ units. We have presented here the results for three point tachyon amplitudes violating winding number by one unit and the pole structure of these expressions has been analysed.

Many open problems remain. On one hand, the computation of four point functions is crucial for many reasons. These expressions are needed to finally answer the question of the closure of the spectrum among the unitary representations. Moreover it would be interesting to see if the free field approximation gives the correct result also for higher point functions. However several subtleties appear in higher point correlators, some of which have been discussed elsewhere.

Closely related to this there is another important issue to be clarified referring to the relevance of the screening operator $\mathcal{S}_-$ and the physical interpretation it can be given in the theory.

Furthermore it would be important to investigate how these issues reflect in the conjectured dual CFT. Additionally the supersymmetric extension of the formalism is an interesting problem in its own.

Note added in proof: after this paper was submitted to hep-th related issues were considered in references [28].

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Erratum

The following list of equations in the text must be modified as:

eq. (2.17)

$$\mathcal{V}_{j,m,\bar{m}} = V_{j,m,\bar{m}} + \ (2j+1)\frac{\Gamma(j-m+1)\Gamma(j+\bar{m}+1)\Gamma(-2j-1)}{\Gamma(-j+\bar{m})\Gamma(-j-m)\Gamma(2j+2)}V_{-1-j,m,\bar{m}}\,;$$

eq. (2.21)

$$R(j) = (2j+1)\left(\frac{\Gamma(1-\rho)}{\Gamma(1+\rho)}\right)^{2j+1}\frac{\Gamma(1+(2j+1)\rho)}{\Gamma\left(1-(2j+1)\rho\right)}\,;$$

eq. (2.59)

$$\begin{aligned} G(j) &= G(-1-j-(k-2))\,, \\ G(j+1) &= \triangle((1+j)\rho)G(j)\,, \\ G(j-k+2) &= (k-2)^{-2j-1}\triangle(1+j)G(j)\,. \end{aligned}$$

This last modification implies the following

eq. (2.60)

$$\tilde{\mathcal{I}}(j_1,j_2,j_3,k) = \frac{1}{k-2}[\pi\tilde{\mu}\triangle(-\tilde{\rho})]^{s_-}D(j_1,j_2,j_3)\,;$$

eq. (2.65)

$$\mathcal{I}(j_1,j_2,j_3,k) = (k-2)[\pi\mu\triangle(-\rho)]^{s_+}D(j_1,j_2,j_3)\,;$$

eq. (3.30)

$$\begin{aligned} \mathcal{A}^{j_1,j_2,j_3}_{m_1,j_2,j_3}(\delta\omega=-1) &= \frac{\Gamma(j_2+j_3-\frac{k}{2}+m_1)^2}{\Gamma(-j_1+m_1)^2\Gamma(2j_2+1)^2\Gamma(2j_3+1)^2}\times \\ &\quad\times(k-2)(\pi\mu\triangle(-\rho))^{s_+}D^{\omega}(j_1,j_2,j_3,k)\,; \end{aligned}$$

eq. (3.32)

$$\begin{aligned} \mathcal{A}^{j_1,j_2,j_3}_{m_1,j_2,j_3}(\delta\omega=-1) &= \triangle(1+j_1-m_1)\triangle(-2j_2)\triangle(-2j_3)\times \\ &\quad\times(k-2)(-\pi\mu\triangle(-\rho))^{s_+}D^{\omega}(j_1,j_2,j_3,k)\,. \end{aligned}$$